

Due Fri

5.2 - Diagonalization

Definition: A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case, the matrix P is said to **diagonalize** A .

Theorem 5.2.1 If A is an $n \times n$ matrix, then A is diagonalizable if and only if A has n linearly independent eigenvectors.

#5 Find a matrix P that diagonalizes A , and check your work by computing $P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{vmatrix} = 0$$
$$\Rightarrow \lambda = \pm 1$$

basis vector for eigenspace for $\lambda = -1$

$$\lambda_1 = -1 \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = 0 \\ x_2 \text{ is free} \end{matrix} \quad \vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow x_1 = \frac{1}{3}x_2$$

We relabel these now: $\vec{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{p}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Form a matrix $P = [\vec{p}_1 \mid \vec{p}_2]$

Here, $P = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$

$$\text{Find } P^{-1}AP: \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

this is diagonal $\rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

- Diagonal entries of $D = P^{-1}AP$ are eigenvalues.
 - Columns of P are the respective eigenvectors.
- The order of these is connected.

What makes this work?

$$P^{-1}AP = D \Rightarrow AP = PD$$

$$\text{Here, } \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array} \right] = \left[\begin{array}{c|c} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array} \right]$$

$$A\vec{p}_1$$

$$A\vec{p}_2$$

$$\lambda_1\vec{p}_1$$

$$\lambda_2\vec{p}_2$$

$$A\vec{p}_1 = \lambda_1\vec{p}_1, \quad A\vec{p}_2 = \lambda_2\vec{p}_2$$

$(A\vec{x} = \lambda\vec{x})$

#17 Compute the matrix A^{10} by first diagonalizing A .

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

Note that if $P^{-1}AP = D$, then $A = PDP^{-1}$.

$$\text{Then } A^k = (PDP^{-1})^k = \underbrace{(PDP^{-1})}_{\text{purple}} \underbrace{(PDP^{-1})}_{\text{purple}} \cdots \underbrace{(PDP^{-1})}_{\text{purple}} \underbrace{(PDP^{-1})}_{\text{purple}}$$
$$= PD^k P^{-1}$$

$$\text{and for } D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}, \quad D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 3 & \\ -2 & \lambda + 1 \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = -3, 2$$

$$\lambda_1 = -3: \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \rightarrow \vec{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2: \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \rightarrow \vec{p}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad P^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix},$$
$$P^{-1}AP = D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$A^{10} = P D^{10} P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 24,234 & -34,815 \\ -23,210 & 35,839 \end{bmatrix}$$

Theorem 5.2.3 If k is a positive integer, λ is an eigenvalue of a matrix A , and \mathbf{x} is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and \mathbf{x} is a corresponding eigenvector.

(not a proof)
 Consider $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = A\lambda\vec{x}$
 $= \lambda(A\vec{x}) = \lambda(\lambda\vec{x})$
 $= \lambda^2\vec{x}$

#9 Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.

- Find the eigenvalues of A .
- For each eigenvalue λ , find the rank of the matrix $\lambda I - A$.
- Is A diagonalizable? Justify your conclusion.

$$\begin{vmatrix} \lambda-4 & 0 & -1 \\ -2 & \lambda-3 & -2 \\ -1 & 0 & \lambda-4 \end{vmatrix} = 0 \Rightarrow (\lambda-3) \begin{vmatrix} \lambda-4 & -1 \\ -1 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-3) [(\lambda-4)^2 - 1] = 0 \Rightarrow (\lambda-3)(\lambda-3)(\lambda-5) = 0$$

$\lambda = 5$: $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(\lambda I - A) = 2$

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{nullity}(\lambda_1 I - A) = 1 = 3 - 2$ $A\vec{x} = \lambda\vec{x}$
 $(\lambda I - A)\vec{x} = \vec{0}$

C dim of eigenspace
 (geometric multiplicity)

$\lambda_2 = 3$:
(alg. mult. 2)

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix}$$

$\text{rank}(\lambda_2 I - A) = 1$
 $\Rightarrow \text{nullity}(\lambda_2 I - A) = 2$

\Rightarrow 7 2 eigenvectors.

$$\lambda_1 = 5 \quad \lambda_2 = 3$$

Total eigenvectors: $1 + 2 = 3$

A is 3x3.

A is diagonalizable because it's 3x3
and we have 3 linearly independent
eigenvectors.

eigenvectors

If λ_0 is an eigenvalue of an $n \times n$ matrix A , then the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0 , and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of λ_0 .

$(\lambda-3)^2(\lambda-5) = 0$
alg mult 2.

Theorem 5.2.4 Geometric and Algebraic Multiplicity

If A is a square matrix, then:

- a) For every eigenvalue of A , the geometric multiplicity is less than or equal to the algebraic multiplicity.
- b) A is diagonalizable if and only if its characteristic polynomial can be expressed as a product of linear factors, and the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

→ # eigenvectors → Characteristic eqn

#14 Find the geometric and algebraic multiplicity of each eigenvalue of the matrix A , and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A , and find $P^{-1}AP$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

$(\lambda-5)^3 = 0 \Rightarrow \lambda = 5$ alg. mult. 3

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

rank is 2 \Rightarrow nullity is 1.
not diagonalizable.

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Theorem 5.2.2

eigenvectors in different eigenspaces are linearly independent.

- a) If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of a matrix A , and if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent set.
- b) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

$(\lambda-2)(\lambda+3)(\lambda-4) = 0$
immediately diagonalizable

$(\lambda-3)^2(\lambda-5) = 0$
we have to check

Similarity

Diagonalizable $\Rightarrow P^{-1}AP = D$ is a diagonal matrix.

Definition: A transformation of the form $A \rightarrow P^{-1}AP$ is called a **similarity transformation**.

Definition: If A and B are square matrices, then we say that **B is similar to A** (or that A and B are **similar matrices**) if there is an invertible matrix P such that $B = P^{-1}AP$.

Example: Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

pf: A is similar to $B \Rightarrow B = P^{-1}AP$, P invertible

$$\Rightarrow \det(B) = \det(P^{-1}AP)$$
$$\Rightarrow \det(B) = \det(P^{-1}) \det(A) \det(P)$$
$$\Rightarrow \det(B) = \frac{1}{\det(P^{-1})} \det(A) \det(P) = \det(A).$$

Any property that is preserved by a similarity transformation is called a **similarity invariant** and is said to be **invariant under similarity**.

Similarity invariants

The following are the same for A and $P^{-1}AP$:

- Determinant
- Trace
- Invertibility
- Characteristic polynomial
- Rank
- Eigenvalues
- Nullity
- Eigenspace dimension

$A\vec{x} = \lambda\vec{x}$ $B\vec{x} = \lambda\vec{x}$ - can be different

Similar same